Formal Summary of Factor Models

Efe Keskin

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This review outlines the evolution and empirical testing of factor models in financial econometrics, with a focus on the Fama-French framework and overall results of the models within the literature.

1 Preliminaries

1.1 Definition of a Formal Factor Model

Suppose that there are k assets and T time periods. Let r_it be the return of asset i in the period t. Then, a generic factor model can be defined as:

$$R_{it} = \alpha_i + \beta_{i1}F_{1t} + \cdots + \beta_{im}F_{mt} + \varepsilon_{it}, \quad t = 1, \ldots, T, \; i = 1, \ldots, K \tag{1}$$

And one has to assume the following properties:

$$E(f_t) = \mu_f, \tag{2}$$

$$\text{Cov}(f_t) = \Sigma_f, \quad \text{an } m \times m \text{ matrix} \tag{3}$$

$$E(\epsilon_{it}) = 0 \quad \text{for all } i \text{ and } t, \tag{4}$$

$$\text{Cov}(f_{jt}, \epsilon_{is}) = 0 \quad \text{for all } j, i, t, \text{ and } s, \tag{5}$$

$$\text{Cov}(\epsilon_{it}, \epsilon_{js}) = \begin{cases} \sigma_i^2, & \text{if } i = j \text{ and } t = s \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

1.2 How Are Factor Models Evaluated?

Before reviewing individual models, it is important to clarify the criteria used to evaluate their empirical and theoretical validity. A successful asset pricing model should meet the following key conditions:

- **Explanatory Power:** It should explain cross-sectional variation in expected returns across assets.
- **Zero Pricing Errors:** The intercepts (α_i) from time-series regressions should be statistically indistinguishable from zero.
- **Low GRS Statistic:** The Gibbons–Ross–Shanken (GRS) test statistic should be low, suggesting no significant mispricing in the model.
- **Economic Interpretation:** The factors used should correspond to well-defined economic risks or macroeconomic sources of uncertainty.
- **Robustness:** The model should perform consistently across time, markets, and portfolio sorts.

These benchmarks help us assess whether improvements like the Fama-French five-factor model or conditional asset pricing approaches genuinely advance the field.

2 Theoretical Foundations of Factor Models

2.1 The CAPM -One Factor Model

The literature accepts that the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) is the beginning of the asset pricing theory. Markowitz's understanding of mean-variance efficiency inspired their understanding, and they applied it in the sense that investors are modelled as *risk-averse* and seek to *minimise variance for a given expected return*. In addition to conceptual application, they assumed:

- **Complete Agreement:** Given the market-clearing asset prices at t-1, investors agree on the joint distribution of asset returns (homogenous expectations).

- Investors can borrow and lend at **a risk-free rate**.

Sharpe and Lintner's version of CAPM was introduced as:

$$x_{i,t} = \alpha_i + \beta_i R_{M,t} + \epsilon_{i,t}, \quad i = 1, \ldots, m; \quad t = 1, \ldots, T \tag{7}$$

Where:

- $x_{i,t} = R_{i,t} - R_f$: excess return of asset i over the risk-free rate at time t,
- $R_{M,t} = R_{M,t} - R_f$: excess return of the market portfolio,
- α_i: intercept (abnormal return) for asset i,
- β_i: market beta of asset i, i.e., sensitivity to market return,
- $\epsilon_{i,t}$: idiosyncratic error term (zero-mean, uncorrelated across assets).

Covariance Matrix of Excess Returns

$$\Sigma_x = \mathrm{Cov}(\mathbf{x}_t) = \sigma_M^2 \boldsymbol{\beta}\boldsymbol{\beta}' + \Psi \tag{8}$$

Where:

- $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)'$: vector of factor loadings,
- $\sigma_M^2 = \mathrm{Var}(R_{M,t})$: variance of the market excess return,
- $\Psi = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_m^2)$: diagonal matrix of idiosyncratic variances.

Total Risk Decomposition for Asset i

$$\mathrm{Var}(x_{i,t}) = \beta_i^2 \sigma_M^2 + \sigma_i^2 \tag{9}$$

Interpretation:

- $\beta_i^2 \sigma_M^2$: component of risk due to systematic (market) exposure,
- σ_i^2: component of risk that is asset-specific and diversifiable.

One can interpret market beta as *a measure of the sensitivity of the asset return* to variation in the market return. And since the model assumes a common tangency point (which becomes linear), it happens to be the slope of the regression of its return on the market return.

The punchline of the model was:

- The market portfolio M must be on the minimum variance frontier if the asset market is clear.
- The risk of the market portfolio can be defined as a measure of the variance of its return, and it is a weighted average of the covariance risks of the assets in it.

Therefore, they were able to argue that **expected returns on all assets are linearly related to their betas**. And they pointed out that **the beta premium is positive**, meaning that the expected return on the market portfolio exceeds the expected return on assets whose returns are uncorrelated with market returns.

2.2 Shortcomings and Motivation for Multi-Factor Model

The empirical tests firmly rejected the model.

- Body of research found out that even though there is a positive relation between beta and average return, it is too flat. (Douglas, 68) (Black, Jensen, Scholes, 72) (Fama & Macbeth, 73).
- The evidence that the relation between beta and average return is too flat is also confirmed in the time-series framework. (Fred & Blume, 72) (Scholes,72)
- The beta is not the only factor that explains the expected returns as one parameter, according to Jensen's Alpha test. He showed significant alpha, especially for low and high betas. (Jensen, 67)

As far as I can observe, empirical contradiction of the Sharpe & Lentner model led to different groups. A group of scholars believes that the contradiction was due to behavioural irregularities of the market. The meaning behind High B/M was that it was mainly due to firms that had fallen during bad times, and a low B/M ratio was seen as a sign of a growth firm. It is quite hard to scientifically prove that the market is pricing the assets in an irrational fashion, even though it is quite easy to show empirically. (Lakonishak, Shleifer, Vishy, 94), (DeBond & Thaler, 87).

Then, another group of scholars got motivated to upgrade the model since the risk definition of the model was single-dimensional and based on a difference equation between the market portfolio and the risk-free rate.

- Ideally, an investor could have intuitively used how their portfolio return covaries with labour income and future investment opportunities as a proxy (Merton,73).
- Fama and French argue in depth that how the investor's wealth at time t might vary with future state variables, including labour income and the prices of consumption goods, should also be included in a model.

These are the pivotal motivators that led to the three, and later five-factor models.

2.3 Fama-French Three-Factor Model (1993)

Fama & French proposed an upgraded model by adding firm size and book-to-market equity. Although not themselves state variables, the higher average returns on small and high book-to-market stocks likely reflect exposure to unidentified risk factors not captured by the market return.

$$E(R_{it}) - R_{ft} = \beta_{iM}\left[E(R_{Mt}) - R_{ft}\right] + \beta_{iS}E(SMB_t) + \beta_{iH}E(HML_t) \tag{10}$$

The authors suggest that the α_i is zero in the following time-series regression:

$$R_{it} - R_{ft} = \alpha_i + \beta_{iM}(R_{Mt} - R_{ft}) + \beta_{iS}SMB_t + \beta_{iH}HML_t + \varepsilon_{it} \tag{11}$$

Meaning that after accounting for the three risk factors, there shouldn't be any systematic unexplained return (alpha) left. Their results were promising (Table 1).

- They reflect the value premium,
- Most have statistically significant t-statistics,

- They show consistent positive mean returns.

Table 1: Annual Value-Weight Dollar Returns for Global Portfolios (1975–1995)

	Market	HB/M	LB/M	**H-LB/M**	HE/P	LE/P	**H-LE/P**	HC/P	LC/P	**H-LC/P**	HD/P	LD/P	**H-LD/P**
Panel A: Excess of T-bill Rate													
Mean	9.60	14.76	7.09	**7.68**	13.66	6.84	**6.82**	13.49	5.89	**7.61**	12.67	7.11	**5.56**
Std.	15.67	16.33	16.13	9.94	17.11	15.59	8.85	17.77	16.05	11.11	16.72	16.09	10.44
t(Mn)	2.74	4.04	1.96	**3.45**	3.57	1.96	**3.45**	3.40	1.64	**3.06**	3.39	1.98	**2.38**
Panel B: Excess of Local Market Return													
Mean	5.16	–	–	**–2.52**	4.06	–2.76	**3.89**	–3.72	**3.07**	–2.49			
Std.	6.95	–	–	3.31	5.95	3.49	6.86	5.47	6.07	4.67			
t(Mn)	3.32	–	–	**–3.40**	3.05	**–3.54**	2.54	**–3.04**	2.26	**–2.38**			

2.4 Shortcomings

The three-factor model proved to be an empirical success. And an intuitive understanding of how a central model can be upgraded. But,

- SMB and HML are empirical constructs. They are not defined by economic theory.
- It's unclear whether they represent risk factors or behavioural irregularity on pricing.
- No link to the returns to macroeconomic variables or state variable, such as intertemporal hedging or recursive utility structure.

The only problem with the model wasn't its empirical nature; the results couldn't explain the momentum as well. Carhart (1997), in his paper "On Persistence in Mutual Fund Performance," showed that adding a momentum factor (UMD = Up Minus Down) to FF3 explains much of the cross-sectional variation in returns missed by the three-factor model.

2.5 Extensions: Five-Factor Model and Beyond

The authors expanded their model to an FF5 in 2015, which is substantial in certain parts. They added profitability and investment as factors, and they supported their idea by the dividend discount model, which was an algebraic way of showing that if one is able to proxy expected investment for a firm and its profitability, the average excess return should be functionally positive. Then the FF5 model was introduced as:

$$R_{it} - R_{ft} = \alpha_i + \beta_i(R_{Mt} - R_{ft}) + s_i SMB_t + h_i HML_t + r_i RMW_t + c_i CMA_t + \varepsilon_{it} \quad (12)$$

In this equation, RMW_t is the difference between the returns on diversified portfolios of stocks with robust and weak profitability, and CMA_t is the difference between the returns on diversified portfolios of the stocks of low and high investment firms, which we call conservative and aggressive. If the exposures to the five factors, β_i, s_i, h_i, r_i, and c_i.

A fair way to assess an asset pricing model to capture whether it captures expected returns, the intercept should be indistinguishable from zero. The FF5 model performed significantly low GRS statistic, which signals that their model is again empirically successful in the first years the article was published. But during and after COVID-19, the GRS statistics weren't as low as they used to be, and the intercept alpha wasn't near zero.

2.6 From FF5 to Conditional Models

Petkova (2006) addresses all of the shortcomings in the model, which are still problematic in the 2015 model of Fama and French 2015. Ralitsa Petkova shown that the FF3 model:

- Treats factor exposures and premiums as constant over time
- Relies on empirically defined factors without economic grounding
- Fails to connect factor behaviour to macroeconomic variables or state dynamics.

Petkova models the betas as functions of lagged macroeconomic variables, allowing them to vary across economic conditions.

$$R_{it+1} - R_{ft+1} = \alpha_i + \beta_i' f_{t+1} + \gamma_i'(Z_t \circ f_{t+1}) + \varepsilon_{it+1} \quad (13)$$

To model conditional betas linearly, she proposes:

$$\beta_{ij,t} = \beta_{ij,0} + \gamma'_{ij} Z_t \tag{14}$$

This structure linearises the time-varying beta idea — betas are not fixed but vary deterministically with observed state variables.

She also proposes that observable factor-mimicking portfolios, such as HML, may serve as proxies for these innovations. So one can write:

$$f_{j,t+1} \propto \eta_{j,t+1} \tag{15}$$

Intuitively, it tells us that HML behaves like a hedge against adverse movements in these states.

Petkova wants to test whether the Fama-French factors (especially HML) proxy for innovations in macroeconomic state variables — i.e., whether these factors behave like shocks to variables that forecast expected returns, such as:

- Dividend yield,
- Default spread,
- Term spread,
- Short-term interest rates.

To do this, she needs to isolate the "innovations" (i.e., unexpected shocks) to these state variables, not just their levels or trends.

$$Z_t = [\mathrm{DP}_t, \mathrm{DEF}_t, \mathrm{TERM}_t, \mathrm{TBILL}_t]' \tag{16}$$

She fits a VAR(1) model:

$$Z_t = AZ_{t-1} + u_t \tag{17}$$

and extracts the innovation vector u_t as the unanticipated shocks to the state variables. These are treated as pricing kernels or relevant risk sources under the ICAPM.

2.7 Key Results of Petkova

2.7.1 HML Behaves Like a Hedging Factor

The HML factor's conditional beta becomes more negative in bad times. This means HML returns go up when the macro environment worsens, which may provide intertemporal hedging.

2.7.2 Better Model Fit

The conditional model with interaction terms reduces pricing errors.

- Cross-sectional R^2 increases, especially in portfolios sorted on book-to-market.
- The alphas (intercepts) are smaller.

2.7.3 Theoretic Support

The model is open to theoretical improvement, but it solves most of the structural problems of Fama-French FF3 and FF5 and applies ICAPM into the Fama-French structure, which may be the best of both worlds.

3 Factor Zoo and ML

The "factor zoo" refers to the overabundance of published "factors". Harvey, Liu and Zhu, in their seminal paper, got motivated to assess the exlplonatory power of the hundreds of empirical factors that has been used in the literature. With over 300 documented factors and likely many more unpublished due to non-significance, the authors argue that conventional significance thresholds are no longer valid because they don't correct for multiple hypothesis testing.

The authors critique the literature of empirical factor modelling:

- Most of the "significant" discoveries are false positives due to extensive data mining.
- The literature rarely publishing replication studies, which exacerbates publication bias.

3.1 Methodolgy, Result and How It Can Be Utilized

They use multiple testing framework in order to asses in depth. After applying statistical corrections for multiple testing such as *Bonferroni (FWER control)*, *Holm's sequential method (FWER)*, they calculate adjusted t-statistic thresholds for discovery over time and project these into the future.

Their main result was **most empirical asset pricing factors are not robust** under proper statistical setup. Due to data snooping and multiple hypothesis testing, the bar for claiming a "significant" factor should be substantially raised— ($t \geq 3.0$) is a more appropriate threshold today.

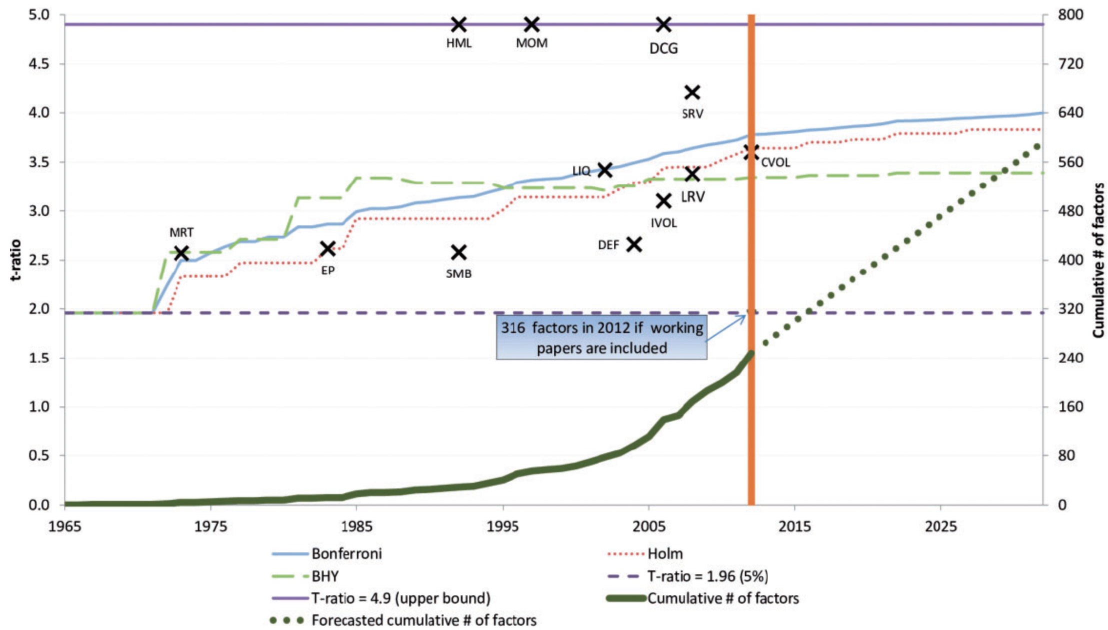

Figure 1: Adjusted t-statistics for Multiple Testing (1965–2032), *from (Harvey, Liu and Zhu, 2015)*

3.2 Via Machine Learning

Machine learning methods has been started to widely use in empirical asset pricing. (Gu, Kelly, and Xiu, 2019), (Xiu,2019), (Kelly, Pruit, and Su, 2019) There are multiple reasons why a machine learning method can outperform applied econometrics, such as "regularization" methods for model selection and mitigation of overfit, but in this case, the high dimensional nature of machine learning methods enhances the flexibility of traditional econometric prediction methods.

(Gu, Kelly, Xiu,2019) reframes asset pricing as a prediction problem — opening the door to tools from statistical learning theory rather than just classical econometrics.

The excess return of stock i at time $t+1$ is modeled as:

$$r_{i,t+1} = E_t\left[r_{i,t+1}\right] + \varepsilon_{i,t+1} = g_\theta\left(z_{i,t}\right) + \varepsilon_{i,t+1} \tag{18}$$

where:

- $z_{i,t}$ is a P-dimensional vector of predictors
- $g_\theta(\cdot)$ is the forecasting function learned via ML
- $\varepsilon_{i,t+1}$ is the unpredictable component

Predictor construction uses:

$$z_{i,t} = x_t \otimes c_{i,t}$$

where $c_{i,t}$ is a 94-dimensional vector of firm characteristics, and x_t includes 8 macroeconomic predictors (e.g., dividend-price ratio, term spread), yielding ∼920 raw features.

3.2.1 Models Compared

- **Linear**: OLS (full and 3-variable: size, value, momentum)
- **Penalized**: Elastic Net, Principal Components Regression (PCR), Partial Least Squares (PLS)
- **Nonlinear**: Generalized linear models (splines), Random Forests, Boosted Trees
- **Deep Learning**: Neural Networks with 1–5 hidden layers (NN1–NN5)

3.2.2 Main Findings

- **Best performer**: Neural Network with 3 hidden layers (NN3)

- **Monthly stock-level $R^2_{\mathbf{oos}}$**:

$$\text{OLS-3: } 0.16\%, \quad \text{Elastic Net: } 0.11\%, \quad \text{NN3: } \mathbf{0.40\%}$$

- **Portfolio timing (S&P 500)**:

$$\text{Sharpe ratio (Buy-and-Hold): } 0.51 \quad \text{vs.} \quad \text{ML (NN3): } \mathbf{0.77}$$

- **Long-short decile strategy Sharpe ratios**:

$$\text{Value-weighted: } \mathbf{1.35}, \quad \text{Equal-weighted: } \mathbf{2.45}$$

3.2.3 Key Insights

- Most predictive variables: momentum, short-term reversal, liquidity, volatility
- Nonlinear models outperform due to ability to capture **interactions** among predictors
- Shallow learning (NN3) outperforms deeper nets (NN4, NN5) due to limited signal-to-noise in returns

While machine learning methods deliver accurate estimates of conditional expected returns, they do not in themselves provide structural economic interpretation. Specifically, the object being estimated in this paper is the conditional expectation:

$$E_t[r_{i,t+1}] = g(z_{i,t}),$$

where $r_{i,t+1}$ denotes the excess return of asset i at time $t+1$, and $z_{i,t}$ is a vector of lagged stock characteristics and macroeconomic variables. Machine learning algorithms is good at approximating $g(\cdot)$ while using nonparametric or high-dimensional techniques, but this function is statistical. One needs theory to develop a better understanding, or a combination of it.

3.2.4 Take-Aways for Econometricians

The top-performing predictors, agreed upon across neural nets, tree models, penalized regressions, and dimension reduction methods, fall into three key categories:

Table 2: Essential Predictive Factors Identified by Gu, Kelly, and Xiu (2020)

Category	**Key Predictive Variables**
1. Momentum and Price Trends	• `mom12m`: 12-month stock momentum • `mom1m`: Short-term reversal (1-month return) • `indmom`: Industry-adjusted momentum • `chmom`: Momentum change / acceleration • `mom36m`: Long-term reversal • `maxret`: Maximum daily return in past month
2. Liquidity and Size	• `mvel1`: Log market equity (firm size) • `dolvol`: Dollar trading volume • `turn`, `std_turn`: Share turnover and its volatility • `ill`: Amihud illiquidity • `baspread`: Bid-ask spread • `zerotrade`: Days with zero trading volume
3. Volatility and Risk Measures	• `retvol`: Total return volatility • `idiovol`: Idiosyncratic volatility • `beta`: Market beta • `betasq`: Beta squared

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